

Marginals with finite Coulomb cost

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Abstract

Given $\rho \in \mathcal{P}(\mathbb{R}^d)$, we prove that, if the concentration of ρ is less than $1/2$, then it has finite 2-particles Coulomb cost. The result is sharp, in the sense that there exists ρ with concentration $1/2$ for which $C(\rho) = \infty$.

1 Introduction

We consider the Coulomb cost function $c: (\mathbb{R}^d)^N \rightarrow \mathbb{R}$ defined as

$$c(x_1, \dots, x_N) = \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|},$$

and the associated optimal transport problem

$$C(\rho) = \inf \left\{ \int_{(\mathbb{R}^d)^N} c(x_1, \dots, x_N) dP(x_1, \dots, x_N) : \right. \\ \left. P \in \mathcal{P}((\mathbb{R}^d)^N), \pi_{\#}^i P = \rho, i = 1, \dots, N \right\},$$

where ρ is a fixed probability measure over \mathbb{R}^d , and π^i is the projection over the i -th factor of $(\mathbb{R}^d)^N$.

Given $\rho \in \mathcal{P}(\mathbb{R}^d)$, the *concentration* of ρ is

$$\mu(\rho) = \sup_{x \in \mathbb{R}^d} \lim_{r \rightarrow 0} \rho(B(x, r)).$$

It is commonly assumed that if $\mu(\rho) < 1/N$ then $C(\rho)$ is finite; however, to our knowledge, the proof of this fact is still to come. In this brief paper we prove this claim in the case $N = 2$. We also prove that this result is sharp, in the sense that, for every $d \geq 1$, there exists $\rho \in \mathcal{P}(\mathbb{R}^d)$ with $\mu(\rho) = 1/2$, and $C(\rho) = \infty$.

2 Notation and sharpness of the result

Using the Lebesgue decomposition, we write

$$\rho = f + \sum_{i=1}^{\infty} c_i \delta_{x_i} + \sigma,$$

where $f \in L^1$, $f \geq 0$, and σ is orthogonal to the Lebesgue measure \mathcal{L}^d . We immediately point out that the singular part of ρ does not play any role, since the trivial plan

$$S(x, y) = \sigma(x) \otimes \sigma(y)$$

has finite Coulomb cost, since it is concentrated over a set of null Lebesgue measure, and $c(x, y)$ is Lebesgue-measurable. For this reason, from now on, ρ will be of the form

$$\rho = f + \sum_{i=1}^{\infty} c_i \delta_{x_i}.$$

2.1 Counterexample with concentration 1/2

For a variable $x \in \mathbb{R}^d$ write $x = (x^1, \dots, x^d)$, and consider the following probability over \mathbb{R}^d :

$$\rho(x) = \delta_0(x^1) \delta_0(x^2) \cdots \delta_0(x^{d-1}) \left(\frac{1}{2} \delta_0(x^d) + \frac{1}{4} \chi_{[-1,1]}(x^d) \right),$$

which has concentration 1/2; let P be any probability measure with marginals equal to ρ . Note that we can assume P to be symmetric, since the cost function c is symmetric. If P had an atom, it should be in the point $(0, 0)$ due to the constraint on the projections; but then its cost will be $+\infty$, so we can assume this is not the case. Define $E = \{0\} \times \cdots \times \{0\} \times [-1, 1] \subset \mathbb{R}^d$, and note that

$$\begin{aligned} P(\{0\} \times E) &= \rho(\{0\}) = \frac{1}{2}, \\ P(E \times \{0\}) &= \rho(\{0\}) = \frac{1}{2}, \end{aligned}$$

and hence P is concentrated over the set $(\{0\} \times E) \cup (E \times \{0\})$, which means

$$P(x, y) = \delta_0(x) \delta_0(y^1) \cdots \delta_0(y^{d-1}) f(y^d) + \delta_0(y) \delta_0(x^1) \cdots \delta_0(x^{d-1}) g(x^d).$$

with $f, g \in L^1(\mathbb{R})$; however, by symmetry, $f = g$. Projecting over the first marginal we get

$$\rho(x) = \|f\|_{L^1} \delta_0(x) + \delta_0(x^1) \cdots \delta_0(x^{d-1}) f(x^d),$$

which implies

$$\|f\|_{L^1} = 1/2, \quad f(t) = \frac{1}{4} \chi_{[-1,1]}(t).$$

Using the symmetry, the Coulomb cost of P is

$$\frac{1}{2} \int_{\mathbb{R}^2} \frac{\delta_0(x) \delta_0(y^1) \cdots \delta_0(y^{d-1}) \chi_{[-1,1]}(y^d)}{|x - y|} dx dy = \frac{1}{2} \int_{[-1,1]} \frac{1}{|t|} dt = +\infty.$$

In the next sections we proceed with the positive part of the proof.

3 Case $\rho = f \in L^1$

Since ρ has no atoms, we may choose three suitable real numbers $0 < r_1 < r_2 < r_3$ such that

$$\rho(B(0, r_1)) = \rho(B(0, r_2) \setminus B(0, r_1)) = \rho(B(0, r_3) \setminus B(0, r_2)) = \rho(\mathbb{R}^d \setminus B(0, r_3)) = \frac{1}{4}.$$

Next we take a measurable function $\phi: \mathbb{R}^d \rightarrow \mathbb{R}^d$, preserving the measure ρ and defined locally such that $\phi(B(0, R_1)) = B(0, r_3) \setminus B(0, r_2)$, $\phi(B(0, r_3) \setminus B(0, r_2)) = B(0, r_1)$, $\phi(B(0, r_2) \setminus B(0, r_1)) = \mathbb{R}^d \setminus B(0, r_3)$, $\phi(\mathbb{R}^d \setminus B(0, r_3)) = B(0, r_2) \setminus B(0, r_1)$. The behaviour of ϕ on the spheres of radii r_1, r_2, r_3 is arbitrary, since they form a set of null Lebesgue measure. A transport plan of finite cost is now given by $P = (\rho, \phi_{\#}\rho)$, since

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x - y|} dP(x, y) = \int_{\mathbb{R}^d} \frac{f(x)}{|x - \phi(x)|} dx \leq \min \left\{ \frac{1}{r_2 - r_1}, \frac{1}{r_3 - r_2} \right\}$$

by the construction of ϕ .

4 Case ρ has a single atom

In this case $\rho = c_1 \delta_{x_1} + f$, with $c_1 < 1/2$ and $f \in L^1$, $f \geq 0$. Note that

$$\int_{\mathbb{R}^d} f = 1 - c_1 > \frac{1}{2} > c_1,$$

and let $r > 0$ such that

$$\int_{\mathbb{R}^d \setminus B(x_1, r)} f = c_1.$$

Consider $P_1(x, y) = \delta_{x_1}(x)f(y)\chi_{\mathbb{R}^d \setminus B(x_1, r)}(y) + \delta_{x_1}(y)f(x)\chi_{\mathbb{R}^d \setminus B(x_1, r)}(x)$, and P_2 a transport plan of finite cost for the L^1 marginal $f\chi_{\overline{B(x_1, r)}}$ (Section 3). $C(P_1)$ is finite, because P_1 has support distant from the diagonal at least r . Moreover, the transport plan $P_1 + P_2$ has marginals equal to ρ :

$$\begin{aligned} \int_{\mathbb{R}^d} P_1(x, y) + P_2(x, y) dy &= c_1 \delta_{x_1}(x) + f(x)\chi_{\mathbb{R}^d \setminus B(x_1, r)}(x) + f(x)\chi_{\overline{B(x_1, r)}}(x) \\ &= c_1 \delta_{x_1}(x) + f(x). \end{aligned}$$

5 Case ρ has a finite number of atoms

Consider $\rho = \sum_{j=1}^n c_j \delta_{x_j} + f$, with $1/2 \geq c_1 \geq \dots \geq c_n > 0$. We claim there exists P_δ transport plan which fixes all the δ 's except c_1 (at most). The remaining mass will be fixed as in Section 4, provided it is strictly small than $\|f\|_{L^1}$.

P_δ will be of the form

$$P_\delta = \sum_{i=1}^n \sum_{j=i+1}^n p_{ij} \{ \delta_{x_i}(x)\delta_{x_j}(y) + \delta_{x_i}(y)\delta_{x_j}(x) \}.$$

with $p_{ij} \geq 0$. Note that

$$C(P_\delta) = \sum_{i=1}^n \sum_{j=i+1}^n \frac{2p_{ij}}{|x_i - x_j|}$$

is finite.

If $c_1 \geq c_2 + \cdots + c_n$, we put $p_{1j} = c_j$ for $j = 2, \dots, n$, $p_{ij} = 0$ elsewhere. In this way, the marginals of P_δ are

$$\rho_\delta = (c_2 + \cdots + c_n)\delta_{x_1} + \sum_{j=2}^n c_j \delta_{x_j}.$$

The remaining mass $(c_1 - c_2 - \cdots - c_n)$ over x_1 may be fixed by f as in Section 4, since

$$c_1 - c_2 - \cdots - c_n < 1 - c_1 - \cdots - c_n = \|f\|_{L^1} \iff c_1 < \frac{1}{2}.$$

In the other case, $c_1 < c_2 + \cdots + c_n$, we claim that P_δ may fix simultaneously all the δ 's. The construction of P_δ is inductive, and based on the following

Lemma 1. *Let $c_1 \geq c_2 \geq \cdots \geq c_n > 0$, with $c_1 < c_2 + \cdots + c_n$. Then there exist $t_2, \dots, t_n \geq 0$ such that*

- (i) $t_2 + \cdots + t_n = c_1$;
- (ii) $c_2 - t_2 \geq \cdots \geq c_n - t_n > 0$;
- (iii) $c_2 - t_2 \leq (c_3 - t_3) + \cdots + (c_n - t_n)$.

Proof. If $c_2 < c_3 + \cdots + c_n$, define

$$t_j = \frac{c_1 c_j}{c_2 + \cdots + c_n}.$$

Note that $c_1 \geq c_2$ and $c_1 < c_2 + \cdots + c_n$ imply $c_3 + \cdots + c_n > 0$.

Clearly $t_2 + \cdots + t_n = c_1$. Moreover, for every $j = 2, \dots, n$

$$c_j - t_j = c_j \left(1 - \frac{c_1}{c_2 + \cdots + c_n} \right) = c_j c_1^*,$$

where $0 < c_1^* < 1$. The properties (ii)-(iii) follow directly from the hypotheses.

Let now be the case $c_2 \geq c_3 + \cdots + c_n$. Define

$$t_2 = \frac{c_1 + c_2 - c_3 - \cdots - c_n}{2},$$

$$t_j = \frac{c_j}{2} \left(1 + \frac{c_1 - c_2}{c_3 + \cdots + c_n} \right) \quad \text{for } j = 3, \dots, n.$$

For every $j = 2, \dots, n$ we have $0 < t_j < c_j$:

$$t_2 \geq \frac{c_1}{2} > 0, \quad t_j \geq \frac{c_j}{2} > 0.$$

$$t_2 < \frac{c_2 + c_3 + \cdots + c_n + c_2 - c_3 - \cdots - c_n}{2} = c_2,$$

$$t_j < c_j \iff \frac{c_1 - c_2}{c_3 + \cdots + c_n} < 1 \iff c_1 - c_2 < c_3 + \cdots + c_n.$$

Proof of (i)

$$\sum_{j=2}^n t_j = t_2 + \frac{c_3 + \cdots + c_n}{2} + \frac{c_1 - c_2}{2} = c_1.$$

Proof of (ii) If $j \geq 3$,

$$c_j - t_j = \frac{c_j}{2} \left(1 - \frac{c_1 - c_2}{c_3 + \dots + c_n} \right) \leq \frac{c_{j+1}}{2} \left(1 - \frac{c_1 - c_2}{c_3 + \dots + c_n} \right)$$

since $c_1 - c_2 < c_3 + \dots + c_n$.

For the other inequality,

$$\begin{aligned} c_2 - t_2 - c_3 + t_3 &= \frac{-c_1 + c_2 + \dots + c_n}{2} + \frac{c_3}{2} \left(-1 + \frac{c_1 - c_2}{c_3 + \dots + c_n} \right) \\ &= \frac{-c_1 + c_2 + \dots + c_n}{2} \left(\frac{-c_3}{c_3 + \dots + c_n} + 1 \right) \\ &= \frac{-c_1 + c_2 + \dots + c_n}{2} \frac{c_4 + \dots + c_n}{c_3 + \dots + c_n} \geq 0 \end{aligned}$$

Proof of (iii)

$$\sum_{j=3}^n (c_j - t_j) = \frac{c_3 + \dots + c_n}{2} + \frac{-c_1 + c_2}{2} = c_2 - t_2.$$

□

Proposition 1. Let $c_1 \geq c_2 \geq \dots \geq c_n > 0$, with $c_1 \leq c_2 + \dots + c_n$. Then P_δ may be chosen such that all the δ 's are fixed.

Proof. By induction on $n \geq 2$.

($n = 2$) In this case we have $c_1 = c_2$, and hence it suffices to take

$$p_{12} = c_1.$$

It is straightforward to check that the marginals of P_δ are equal to $\rho_\delta = c_1 \delta_{x_1} + c_2 \delta_{x_2}$.

($n - 1$) \implies (n) If $c_1 = c_2 + \dots + c_n$ there is nothing to prove, simply take $p_{1j} = c_j$ for $j = 1, \dots, n$. If the inequality is strict, take t_2, \dots, t_n given by Lemma 1, and consider $\tilde{c}_j = c_j - t_j$. They satisfy the hypotheses, and hence by induction there exists $P_\delta^{(n-1)}$ which fixes them. Now enlarge to P_δ letting

$$\begin{aligned} p_{ij}^{(n)} &= p_{ij}^{(n-1)} \quad \text{for } i = 2, \dots, n, j > i; \\ p_{1j}^{(n)} &= t_j \quad \text{for } j > 1. \end{aligned}$$

It is simple to conclude: if $i \geq 2$,

$$\sum_{j>i} p_{ij}^{(n)} + \sum_{j<i} p_{ji}^{(n)} = p_{1i}^{(n)} + \sum_{1<j<i} p_{ji}^{(n-1)} + \sum_{j>i} p_{ij}^{(n-1)} = t_i + (c_i - t_i) = c_i,$$

while by Lemma 1

$$\sum_{j=2}^n p_{1j}^{(n)} = \sum_{j=2}^n t_j = c_1.$$

□

6 Case ρ has an infinite (countable) number of atoms

The main difficulty which may arise in this case is a topological issue, since there is not any uniform bound for the values $\frac{1}{|x_i - x_j|}$, when $i \neq j$, $i, j \in \mathbb{N}$. For this reason the construction is a bit more involved. First we fix $\ell_1, \ell_2 \geq 2$ distinct integers such that

$$c_{\ell_1} > c_{\ell_1+1}, \quad c_{\ell_2} > c_{\ell_2+1}.$$

Take H to be the affine hyperplane orthogonal to $x_{\ell_2} - x_{\ell_1}$, and passing through $\frac{x_{\ell_1} + x_{\ell_2}}{2}$; H provides a partition of \mathbb{R}^d into two disjoint regions E_1, E_2 , with $x_{\ell_k} \in E_k$ for $k = 1, 2$. We arbitrarily say that $H \subset E_1$. This partition induces $\mathbb{N} = I \sqcup J$, where

$$I = \{i \in \mathbb{N} \mid x_i \in E_1\}, \quad J = \{j \in \mathbb{N} \mid x_j \in E_2\}.$$

Since $c_1 < 1/2$ and the series $\{c_i\}$ is convergent, we may find a natural number K , greater than $\max\{\ell_1, \ell_2\}$, such that, set

$$\beta_1 := \sum_{\substack{i \in I \\ i > K}} c_i, \quad \beta_2 := \sum_{\substack{j \in J \\ j > K}} c_j,$$

we have:

- $c_{\ell_1} - \beta_2 > c_{\ell_1+1}$;
- $c_{\ell_2} - \beta_1 > c_{\ell_2+1}$;
- $c_1 + \beta_1 + \beta_2 < 1/2$.

We define a first transport plan P as

$$\begin{aligned} P = & \sum_{\substack{i \in I \\ i > K}} c_i (\delta_{x_i}(x) \delta_{x_{\ell_2}}(y) + \delta_{x_i}(y) \delta_{x_{\ell_2}}(x)) \\ & + \sum_{\substack{j \in J \\ j > K}} c_j (\delta_{x_j}(x) \delta_{x_{\ell_1}}(y) + \delta_{x_j}(y) \delta_{x_{\ell_1}}(x)). \end{aligned}$$

Clearly, P is symmetric; its marginals are

$$\sigma(x) = \int_{\mathbb{R}^d} P(x, y) dy = \beta_2 \delta_{x_{\ell_1}}(x) + \beta_1 \delta_{x_{\ell_2}}(x) + \sum_{\substack{i \in I \\ i > K}} c_i \delta_{x_i}(x) + \sum_{\substack{j \in J \\ j > K}} c_j \delta_{x_j}(x).$$

Moreover, P has finite cost, since every x_i , $i \in I$, is at least $|x_{\ell_1} - x_{\ell_2}|/2$ distant from x_{ℓ_2} , and every x_j , $j \in J$, is at least $|x_{\ell_1} - x_{\ell_2}|/2$ distant from x_{ℓ_1} by construction.

It remains to find a finite-cost transport plan for the marginal $\rho - \sigma$, which reads

$$\rho - \sigma = f + \sum_{i=1}^K \tilde{c}_i \delta_{x_i},$$

where $\tilde{c}_{\ell_1} = c_{\ell_1} - \beta_2$, $\tilde{c}_{\ell_2} = c_{\ell_2} - \beta_1$, $\tilde{c}_i = c_i$ for every other $i \leq K$. Note that the \tilde{c}_i 's are decrescent. If $\tilde{c}_1 \leq \tilde{c}_2 + \dots + \tilde{c}_K$ we conclude as in Section 5, second part. On the other hand, suppose that $\tilde{c}_1 > \tilde{c}_2 + \dots + \tilde{c}_K$, and note that $c_1 = \tilde{c}_1$ thanks to the hypothesis $\ell_1, \ell_2 \geq 2$. Then

$$\begin{aligned} \|f\|_{L^1} &= 1 - \sum_{i=1}^{\infty} c_i = 1 - \left(\sum_{i=1}^K c_i \right) - \beta_1 - \beta_2 \\ &= 1 - 2c_1 - 2\beta_1 - 2\beta_2 + c_1 - \sum_{i=2}^K \tilde{c}_i \\ &> c_1 - \sum_{i=2}^K \tilde{c}_i, \end{aligned}$$

and hence we can conclude as in Section 5, first part.

7 Conclusion

We believe that the technique developped in this proof may lead to an analogous theorem for any N , to which the author is already working. It is worth noting also that the positive part of the proof works in the same way for any kind of repulsive cost, while the sharpness is related to the divergence of the integral of $1/x$ near zero.

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